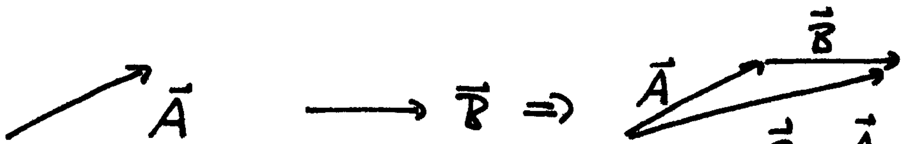
Vectors and Vector Calculus



$\vec C = \vec A + \vec B = \vec B + \vec A$

Simple def: Magnitude and direction

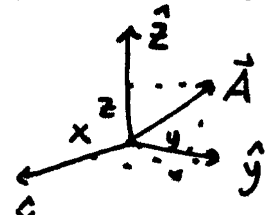
Given a coordinate system we can determine a set of coordinates

$$\vec r = (x_1, \dots, x_d) \quad (d \text{ dimensions.})$$

By introducing unit vectors in a (Cartesian) coordinate system we may write:

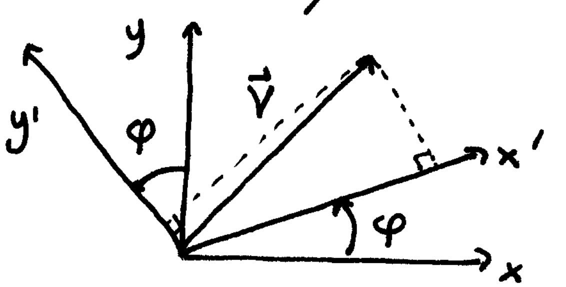
$$\vec r = x\,\hat x + y\,\hat y + z\,\hat z$$

linearly independent basis



Magnitude of a vector: $|\vec r| = r = \sqrt{x^2+y^2+z^2}$ by the Pythagorean theorem.

A more sophisticated and general view of vectors defines them by their transformation properties under rotations of the coordinate system:



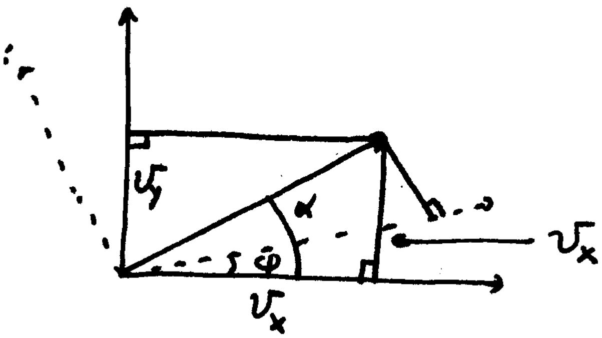
$\vec V = v_x\,\hat x + v_y\,\hat y$ or

$\vec V = v_x'\,\hat x' + v_y'\,\hat y'$ but this is the <u>same</u> vector!

Thus,

$$\boxed{\begin{aligned} v_x' &= v_x \cos\varphi + v_y \sin\varphi \\ v_y' &= -v_x \sin\varphi + v_y \cos\varphi \end{aligned}}$$

Let's check the first line:



$$\alpha = \arctan(v_y/v_x)$$

$$v_x' = |\vec{v}|\cos(\alpha-\varphi) = |\vec{v}|\left[\cos\alpha\cos\varphi + \sin\alpha\sin\varphi\right]$$

and $\cos(\alpha) = \dfrac{v_x}{|\vec{v}|}$, $\sin(\alpha) = \dfrac{v_y}{|\vec{v}|}$ $\Rightarrow$

$$v_x' = v_x\cos\varphi + v_y\sin\varphi \quad \checkmark$$

Thus the representation of the vector $\vec{v}$ transforms under rotations by

$$\begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

Note: one angle but four matrix elements $\Rightarrow$ inelegant!

The transformation matrix A_{ij} is really the set of "direction cosines" or dot products between the old and new coordinate axes.

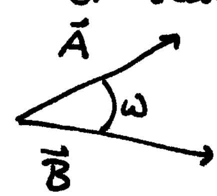
$$A_{ij} = \begin{pmatrix} \hat{x}\cdot\hat{x}' & \hat{y}\cdot\hat{x}' \\ \hat{x}\cdot\hat{y}' & \hat{y}\cdot\hat{y}' \end{pmatrix}$$

row

column

where we have introduced a "dot" or scalar product of two vectors:

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\omega$$



More on both matrix multiplication and dot products soon.

Note that we can also multiply a vector by a scalar:

$a\vec{A} = (ax_1, ax_2, ax_3, \ldots; ax_d)$ when $\vec{A} = (x_1, \ldots, x_d)$

So far we can add vectors: $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ assoc.

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ comm.

And multiply by a scalar: $\alpha\vec{a}$

Now, we look at scalar multiplication of vectors.

1st for orthogonal unit vectors

$$\left.\begin{array}{lll} \hat{x}\cdot\hat{x} = 1, & \hat{x}\cdot\hat{y} = 0, & \hat{x}\cdot\hat{z} = 0 \\ \hat{y}\cdot\hat{x} = 0, & \hat{y}\cdot\hat{y} = 1, & \hat{y}\cdot\hat{z} = 0 \\ \hat{z}\cdot\hat{x} = 0 & \hat{z}\cdot\hat{y} = 0, & \hat{z}\cdot\hat{z} = 1 \end{array}\right\}$$

We can write $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

Kronecker delta

$\delta_{ij} = 0$ if $i \neq j$

1 otherwise.

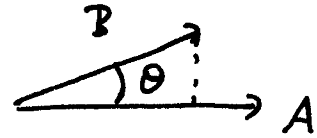
Since we can write $\vec{A} = \sum_{i=1}^{d} a_i \hat{e}_i$, $\vec{B} = \sum_{j=1}^{d} b_j \hat{e}_j$

$$\vec{A}\cdot\vec{B} = \sum_{i,j=1}^{d} a_i b_j \,\hat{e}_i \cdot \hat{e}_j = \sum_{i=1}^{d} a_i \cdot b_i$$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = \sum_{i=1}^{d} a_i b_i \qquad \text{Scalar Product.}$$

Is this the same as our previous definition?

Put $\vec{A}$ along the $\hat{x}$ axis:



$$\Rightarrow \quad A_i = \delta_{i1}\,|\vec{A}|$$

$$B_1 = |\vec{B}|\cos\theta \quad \Rightarrow \quad \vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta \quad \checkmark$$

Why do we call this a <u>scalar</u> product?

Ans Consider the change in $\vec{A}\cdot\vec{B}$ if we rotation the coordinate system.

$$\vec{A}'\cdot\vec{B}' = \sum_{i,k}\sum_{j} \underbrace{R_{ij}A_j}_{A_i'}\;\overbrace{R_{ik}B_k}^{B_i'}$$

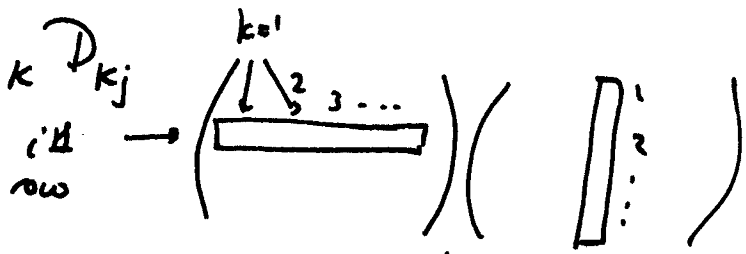
Transpose: switch rows and colums.

$$\text{look at} \quad \sum_i R_{ij}R_{ik} = \sum_i R^T_{ji}R_{ik}$$

same as R_{ij}

Now

$$\sum_k C_{ik}D_{kj}$$



This is just matrix multiplication. j th column

In fact

$$\sum_k C_{ik}D_{kj} = (CD)_{ij} \leftarrow \text{The } (i,j) \text{ element of the}$$

product of matrices C and D.

so $\sum_i R^T_{ji} R_{ik} = (R^T R)_{jk}$

Let's check this matrix product: from page 2 $\Rightarrow$

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

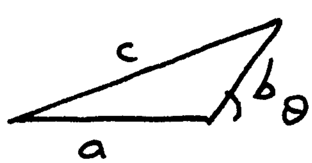
The identity matrix $\Leftrightarrow$ the matrix I for which: $\vec{A} \cdot I = I \cdot \vec{A} = \vec{A}$. Note $I_{ij} = \delta_{ij}$

so

$$\vec{A}' \cdot \vec{B}' = \sum_{ij} A_j B_k \delta_{jk} = \sum_i A_i B_i = \vec{A} \cdot \vec{B}$$

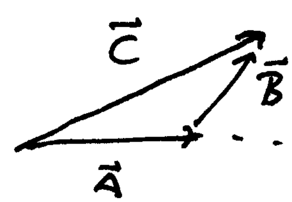
The *scalar* product is invariant under rotation. This is what we mean by the adjective "scalar."

As an aside, we can easily prove the "law of cosines"



$c^2 = a^2 + b^2 + 2ab\cos\theta$; when $\theta = \pi/2$ this is just the Pythagorean theorem.

Write this as vectors



$|\vec{A}| = a$ etc.

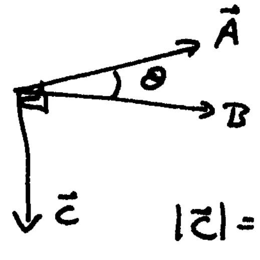
$$\vec{C} = \vec{A} + \vec{B} \Rightarrow \vec{C} \cdot \vec{C} = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = a^2 + b^2 + 2\vec{A} \cdot \vec{B}$$

or $\quad c^2 = a^2 + b^2 + 2ab\cos\theta$

We can also form a vector product of two vectors.

Vector or Cross-Product.

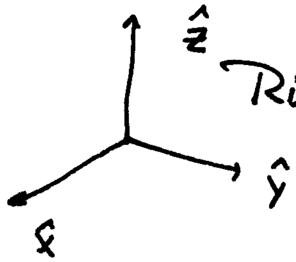
$\vec{C} = \vec{A} \times \vec{B}$



Right Hand Rule.

$|\vec{C}| = AB \sin\theta$

Anticommutation: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$



Right handed coordinate system $\Rightarrow$

$\hat{x} \times \hat{y} = \hat{z}$ $\quad \hat{y} \times \hat{z} = \hat{x}$

$\hat{x} \times \hat{z} = -\hat{y}$

We can write this in components using the antisymmetric tensor

ε_{ijk} (three dimensional) [Levi-Civita Symbol]

$\varepsilon_{123} = 1$ (Right handed coordinate system)

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } \text{" " odd " "} \end{cases}$$

Note: ε is zero for two repeated indices

e.g. $\varepsilon_{132} = -1$, $\varepsilon_{312} = +1$ ← a "cyclic perturbation"

Then

$$\underbrace{\boxed{C_i = \varepsilon_{ijk} A_j B_k}}_{}$$

why? check w/ the vectors $\hat{x}$ $\hat{y}$ and $\hat{z}$.

Using the Einstein summation convention

We can write the same thing in terms of the determinant:

$$\vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Note: $C_1 = \varepsilon_{1jk} A_j B_k = \varepsilon_{123} A_y B_z + \varepsilon_{132} A_z B_y$

$$= A_y B_z - A_z B_y$$

checks w/ determinant above.

We will not check that $\vec{C}$ is a vector now, but come back to this question later. The answer is that $\vec{C}$ is not a true vector, but, in 3D it is close. Will define a new term for this object - A pseudovector.

The triple scalar product: $\vec{A} \cdot (\vec{B} \times \vec{C})$

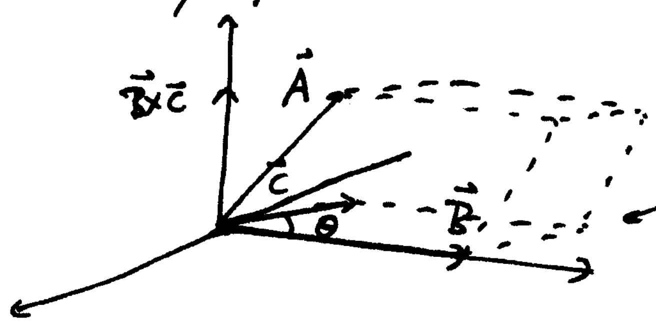
Using the determinant form of the crossproduct we see that

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Do you see why? write it out and you will!

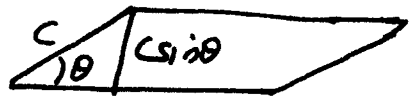
Equal to the volume of the parallelepiped defined by $\vec{A}$, $\vec{B}$, and $\vec{C}$.

To see why, put $\vec{B}$ and $\vec{C}$ in the xy plane:

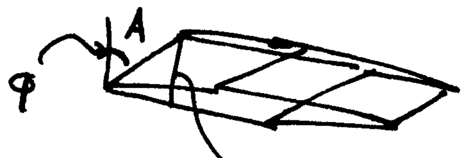


← area of the base

$BC\sin\theta$



looking down at the xy plane.



$A\cos\varphi$ or $\vec{A}\cdot(\vec{B}\times\vec{C}) =$ volume $ABC\cos\varphi\sin\theta$ ✓

We can also write the triple vector product

$$\left[\vec{A}\times(\vec{B}\times\vec{C})\right]_\alpha = A_i B_j C_k \left[\varepsilon_{\alpha i l}\,\varepsilon_{l j k}\right]$$

What is $\varepsilon_{\alpha i l}\,\varepsilon_{l j k}$?

$$= \varepsilon_{\alpha i 1}\,\varepsilon_{1jk} + \varepsilon_{\alpha i 2}\,\varepsilon_{2jk} + \varepsilon_{\alpha i 3}\,\varepsilon_{3jk} = \;?$$

It must be that (αi) pair up w/ (jk) if

if $\alpha = j$, $i = k$ we get $\varepsilon_{jk1}\,\varepsilon_{1jk} = (+1)(+1) = +1$

e.g. 231 123

$\alpha = k$, $i = j$ we get $\varepsilon_{kj1}\,\varepsilon_{1jk} = (-1)(+1) = -1$

321 123

So: $\varepsilon_{\alpha\beta l}\,\varepsilon_{ljk} = \delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}$

Thus,

$$[\vec{A}\times\vec{B}\times\vec{C}]_\alpha = A_i B_j C_k\,[\delta_{\alpha j}\delta_{ik} - \delta_{\alpha k}\delta_{ij}]$$
$$= B_\alpha(\vec{A}\cdot\vec{C}) - C_\alpha(\vec{A}\cdot\vec{B})$$

or

$\vec{A}\times\vec{B}\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$ This is sometimes called the "bac-cab" rule, but you might as well understand how to derive it so you don't have to remember silly names.

Vector Calculus: The Gradient $\vec{\nabla}$

Making a vector out of a scalar function:

Let $\psi(x,y,z)$ be a scalar function. In other words, under rotations of our coordinate system we get the same value,

i.e. $\varphi'(x',y',z') = \varphi(x,y,z)$

Consider the partial derivatives of this function

$$\frac{\partial\varphi'}{\partial x_i'} = \frac{\partial\varphi}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'}\frac{\partial\varphi}{\partial x_j} \quad \text{(Chain rule)}$$

what is $\frac{\partial x_j}{\partial x_i'}$? Well, $x_i' = R_{ij}\,x_j$ ← Rotation matrix

so $\frac{\partial x_i'}{\partial x_j} = R_{ij} \Rightarrow \frac{\partial x_j}{\partial x_i'} = R_{ij}$ as well.

so $\frac{\partial \varphi'}{\partial x_i'} = R_{ij} \frac{\partial \varphi}{\partial x_j} \Rightarrow \frac{\partial \varphi}{\partial x_j}$ is a vector!

We will call it the gradient: $\vec{\nabla}\varphi$

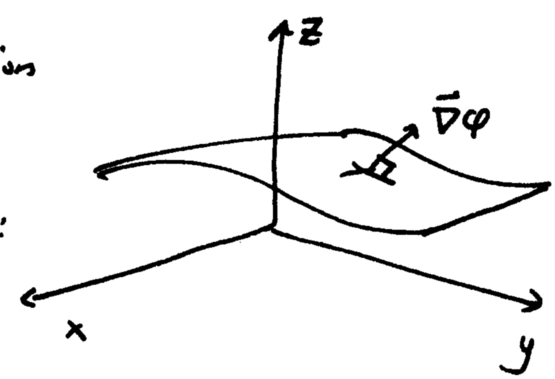
$$\vec{\nabla}\varphi = \hat{x}\, \partial_x \varphi + \hat{y}\, \partial_y \varphi + \hat{z}\, \partial_z \varphi$$

A geometrical interpretation:

$$\vec{\nabla}\varphi \cdot d\vec{x} = dx\, \partial_x \varphi + \partial_y \varphi\, dy + \partial_z \varphi\, dz$$

$$= d\varphi$$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{change in} \\ \text{the function}}}{d\varphi} = \vec{\nabla}\varphi \cdot \underset{\substack{\uparrow \\ \text{change in position}}}{d\vec{x}}$$



Consider a surface of constant φ:

<u>It must be normal to $\vec{\nabla}\varphi$</u>

so when you move along the surface $d\vec{x}_\perp$, $d\varphi = 0$.

Example:

$$\varphi = f(r) \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\vec{\nabla}\varphi = \hat{e}_i\, \partial_i f(r) = \hat{e}_i \frac{df}{dr}\frac{\partial r}{\partial x_i} = \hat{e}_i \frac{df}{dr} \frac{1}{\sqrt{x^2+y^2+z^2}}\, x_i$$

$\Rightarrow \vec{\nabla}\varphi = \frac{\vec{x}}{|\vec{x}|}\frac{df}{dr} \Leftarrow$ surfaces of constant are <u>spherical</u>

as you would expect!

As a special case $f(r) = r^n$

$$\vec{\nabla}\varphi = \hat{r}\, n r^{n-1} \leftarrow \text{more on this one later....}$$

Derivatives of a vector field: The divergence and the curl.

Treating $\vec{\nabla}$ as a differential vector operator, we should be able to define $\vec{\nabla}\cdot\vec{V}$ on a vector field $\vec{V}(\vec{x})$

$$\vec{\nabla}\cdot\vec{V} = \partial_i V_i$$

e.g. $\vec{\nabla}\cdot\vec{r} = \partial_i x_i = 3$ (in three dimensions)

$$\vec{\nabla}\cdot(\hat{r}\, n r^{n-1}) = \vec{\nabla}\cdot(\vec{r}\, n r^{n-2})$$

$$= \partial_i (x_i\, n r^{n-2}) = 3 n r^{n-2} + x_i\, n(n-2) r^{n-3} \frac{\partial r}{\partial x_i}$$

and $\dfrac{\partial r}{\partial x_i} = \dfrac{1}{\sqrt{}} \dfrac{1}{2} 2x_i = \dfrac{x_i}{r}$

$$\Rightarrow \vec{\nabla}\cdot(\hat{r}\, n r^{n-1}) = r^{n-2}[3n + n^2 - 2n]$$

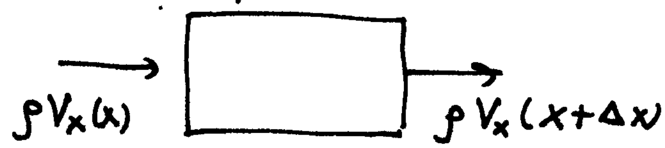
$$= n r^{n-2}[n+1]$$

so $\vec{\nabla}\cdot\vec{\nabla}(r^n) = n r^{n-2}(n+1)$ Something mysterious happens when $n = -1$ ie for $f \sim 1/r$. More on this later.

To try to get a physical feel for the divergence consider the following...

Consider $\vec{\nabla}\cdot(\rho\vec{v})$

velocity of (say) a gas carrying a density $\rho(x,y,z)$ of particles!



say the flow is in the $\hat{x}$ direction

The rate of accumulation of ρ in the volume $dx\,dy\,dz$ is

$$dy\,dz\left\{\rho V_x(x) - \rho(x+\Delta x)\,V_x(x+\Delta x)\right\} = -dx\,dy\,dz\left[\partial_x(\rho V_x)\right]$$

In general, we need to consider the flow in the y and z directions:

$$\Longrightarrow \quad \frac{\partial n}{\partial t} = dx\,dy\,dz\left\{-\partial_x(\rho V_x) - \partial_y(\rho V_y) - \partial_z(\rho V_z)\right\}.$$

$n = \Delta V \rho$ counts the particles.

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{V})$$ Conservation of particles.

The curl $\vec{\nabla}\times$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \end{vmatrix} \longleftarrow \text{But be careful!}$$

1) $\vec{\nabla}\times\vec{V} \neq -\vec{V}\times\vec{\nabla}$ These are two very different animals.

(axial) vector   operator.

2) expand along the top row!

3) Note that differential operators like $\vec{L} = -i\,\vec{r}\times\vec{\nabla}$ do not behave like vectors under the cross product

i.e. $\vec{L}\times\vec{L} = i\vec{L}$ ← as you probably remember from the commutation relations of angular momentum.

Perhaps it is safer to write this in terms of components.

$(\vec{\nabla}\wedge\vec{V})_\alpha = \varepsilon_{\alpha\beta\gamma}\,\partial_\beta V_\gamma$. From this we can find the curl of a vector field times a scalar field.

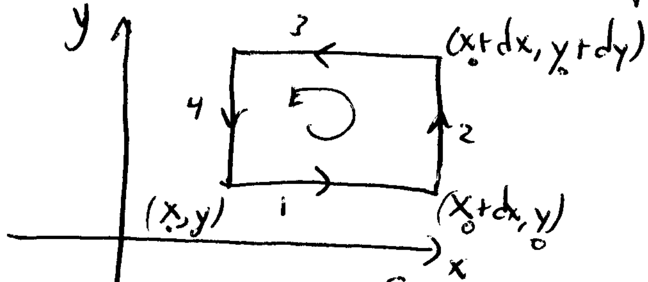
$$\left[\vec{\nabla}\times(f\vec{W})\right]_\alpha = \varepsilon_{\alpha\beta\gamma}\,\partial_\beta\,(fW_\gamma)$$

$$= \varepsilon_{\alpha\beta\gamma}\left[(\partial_\beta f)W_\gamma + f\,\partial_\beta W_\gamma\right]$$

$$= \vec{\nabla}f\times\vec{W} + f\,\vec{\nabla}\wedge\vec{W}$$

The curl and the circulation of a vector field

To get a better understanding of the curl let's compute the integral $\oint \vec{V}\cdot d\vec{s}$ around a closed loop. To make this simple, we can take a loop of infinitesimal extent:



$$C = (\text{circulation}) = \int_1 V_x\, dx + \int_2 V_y\, dy + \cdots$$

writing $V_\alpha(x,y) = V_\alpha(x_0, y_0) + \dfrac{\partial V_\alpha}{\partial x} dx + \dfrac{\partial V_\alpha}{\partial y} dy + \cdots$

So we can expand the circulation C as

$$dC = V_x(x_0, y_0)\, dx + V_y(x_0+dx, y_0)\, dy - V_x(x_0+dx_0, y_0+dy_0)\, dx$$

$$- V_y(x_0, y_0+dy)\, dy = dx\left[-\frac{\partial V_x}{\partial y} dy\right] + dy\left[\frac{\partial V_y}{\partial x} dx\right] + \mathcal{O}(dx^2, dy^2)$$

$$dC = dx\,dy\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

circulation per area is $(\vec{\nabla}\times\vec{V})_z$ ← We will come back to this point

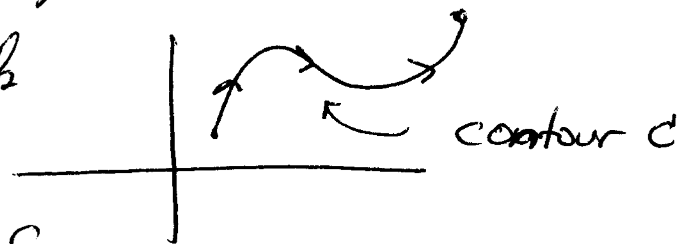
Just some names: They will mean more to us later.

$\vec{\nabla}\times\vec{V} \iff \vec{V}$ is irrotational

$\vec{\nabla}\cdot\vec{V} \iff \vec{V}$ is solenoidal.

1.10 Vector Integration

Line Integrals

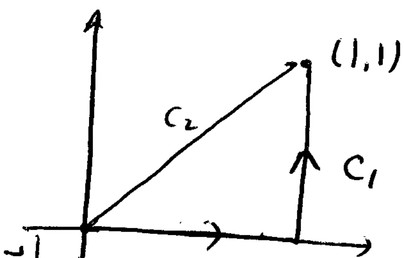


$$\int_C \phi \, d\vec{s} \quad , \quad \int \vec{V} \cdot d\vec{s}$$

this is a vector ↑   ↑ this is a scalar

A couple of examples.

$\vec{V} = \hat{x}\, 3x + \hat{y}$ What is

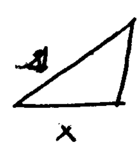


$$\int_{C_1} d\vec{s} \cdot \vec{V} = \int_0^1 dx\, \hat{x} \cdot \vec{V}\Big|_{y=0} + \int_0^1 dy\, \hat{y} \cdot \vec{V}\Big|_{x=1}$$

$$= \int_0^1 3x\, dx + \int_0^1 dy = \frac{3}{2} + 1 = 5/2$$

$$\int_{C_2} d\vec{s} \cdot \vec{V} =$$

$$d\vec{s} = \underbrace{\frac{\hat{x} + \hat{y}}{\sqrt{2}}}_{\text{unit vector}} \overbrace{ds}^{\text{scalar}} \qquad 0 \le s \le \sqrt{2}$$



$$x = s/\sqrt{2}$$

$$\vec{V} \cdot d\vec{s} = \left(\frac{3x}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) ds = \left(\frac{3}{2} s + \frac{1}{\sqrt{2}}\right) ds$$

$$\int_0^{\sqrt{2}} \vec{V}\cdot d\vec{s} = \frac{3}{2}\frac{s^2}{2}\Big|_0^{\sqrt{2}} + \frac{1}{\sqrt{2}} s\Big|_0^{\sqrt{2}} = \frac{3}{2} + 1 = 5/2$$

same answer!

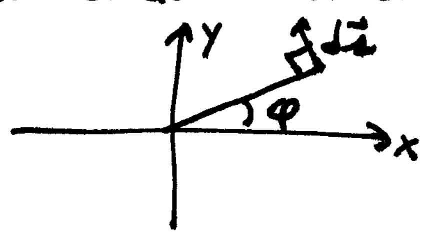
Or, we could have written this as ... $d\vec{s} = (\hat{x} + \hat{y})\,ds$ and let s range from 0 to 1. This just means we traversed the path "faster". You can check that this works.

Watch the dot product in these expressions!

$\oint d\vec{s} = 0$ for a closed path! do you see why?

But $\oint d\vec{s}\cdot\vec{w} \neq 0$ if we take $\vec{w} = x\hat{y} - y\hat{x}$

To test this we can take the integration path to be the unit circle traversed in a counter clockwise direction



$$d\vec{s} = d\left(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}\right)$$

$$d\vec{s} = d\varphi\left[-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}\right]$$

and on the circle, $\vec{w} = \cos\varphi\,\hat{y} - \sin\varphi\,\hat{x} \Rightarrow$

$$\vec{w}\cdot d\vec{s} = d\varphi\left(\cos^2\varphi + \sin^2\varphi\right) = d\varphi \Rightarrow$$

$$\oint \vec{w}\cdot d\vec{s} = 2\pi \qquad \text{But } \oint d\vec{\varphi} = \int_0^{2\pi} d\varphi\left[-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}\right]$$

$= 0.$

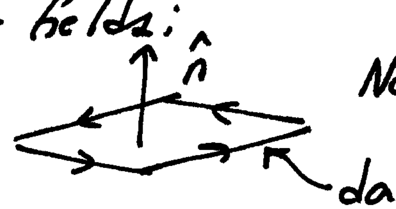
You may have noticed that in the last example we found that the integral of $\vec{w}$ gave zero for a closed path but this was not true for the example before that.

We will see why that was the case by looking at a more general theorem a bit later.

First, we need to look at surface integrals and volume integrals of vector fields:

$$d\vec{a} = \hat{n}\, da$$

unit normal



Normal direction chosen by the RH rule.

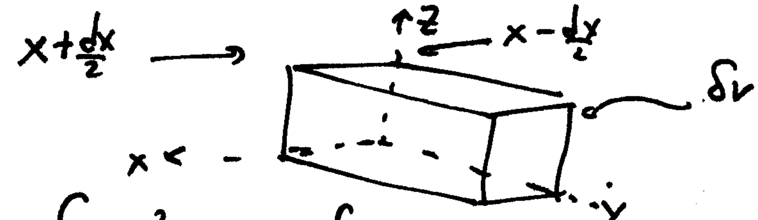
The relation between the gradient, divergence, and curl and integrals over surfaces.

$$\vec{\nabla}\cdot\vec{V} = \lim_{\delta v \to 0} \frac{\int \vec{V}\cdot\hat{n}\, da}{\delta v}$$ ← See page 12.

$$\vec{\nabla}\times\vec{V} = \lim_{\delta v \to 0} \frac{\int da\, \hat{n}\times\vec{V}}{\delta v}$$

$$\vec{\nabla}\varphi = \lim_{\delta v \to 0} \frac{\int \varphi\, d^2a\, \hat{n}}{\delta v}$$

Let's work out the last one in more detail



$$\int \varphi\, d\vec{a} = -\hat{x}\int dy\, dz\left(\varphi - \frac{\partial_x \varphi}{2}\, dx\right) + \hat{x}\int dy\, dz\left(\varphi + \frac{\partial_x \varphi}{2}\, dx\right) +$$

outward normal at the back

outward normal at the front.

+ similar terms with $x \to y \to z$,

So we get:

$$\int \varphi \, d^2\vec{a} = \hat{x} \int dx\,dy\,dz\,(\partial_x \varphi) + \hat{y} \int_{\delta V} dx\,dy\,dz\,(\partial_y \varphi) + \cdots$$

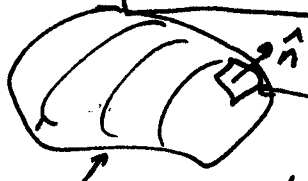
or for a small enough volume, so we can take $\vec{\nabla}\varphi$ to be a constant vector.

$$\int \varphi \, d^2\vec{a} = \vec{\nabla}\varphi \, (\delta V) \quad \checkmark$$

You should try the curl equation yourself to see if you have it down.

Gauss's Theorem:

$$\boxed{\oint_S \vec{V} \cdot d^2\vec{a} = \int_V \vec{\nabla} \cdot \vec{V} \, d^3x}$$



$\hat{n}$ outward normal.

surface S bounds volume V

as a proof, just use our above results for infinitesimal volumes and break the big volume V up into contiguous infinitesimal ones...

Side Note

One particularly useful application of Gauss' theorem is found in using it to understand Green's functions.

Consider the following identity:

Side Note continues

Let u and v be two functions then

$$\vec{\nabla}\cdot(u\vec{\nabla}v) = u\nabla^2 v + (\vec{\nabla}u)\cdot(\vec{\nabla}v)$$ do you see why?

$$\vec{\nabla}\cdot(v\vec{\nabla}u) = v\nabla^2 u + (\vec{\nabla}v)\cdot(\vec{\nabla}u)$$

subtract one from the other and integrate over a closed volume bounded by S.

$$\int_V \vec{\nabla}\cdot[u\vec{\nabla}v - v\vec{\nabla}u]\,d^3x = \int_V d^3x\,[u\nabla^2 v - v\nabla^2 u]$$

↓ Gauss's Theorem!

$$= \int_S [u\vec{\nabla}v - v\vec{\nabla}u]\cdot\hat{n}\,da^2$$

This doesn't look so useful at first but we'll see why this is useful later. To get the idea now imagine that v "undoes" the Laplacian ∇^2. In other words

$$\nabla^2 v(\vec{x}-\vec{x}') = \delta(\vec{x}-\vec{x}') \leftarrow \text{Dirac delta function}$$

{We will see that this isn't so strange afterall →

$$v(\vec{x}-\vec{x}') = \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}$$ will work in three dimensions}

Now suppose that u obeys a partial differential equation

$$\nabla^2 u = \rho(x) \leftarrow \text{some prescribed function.}$$

The problem is: What is u given some bounded ρ. In other words say $\rho \to 0$ at large distances from the origin!

Then, taking a large enough V we can make the integral over the surface S vanish (at $V \to \infty$). Then

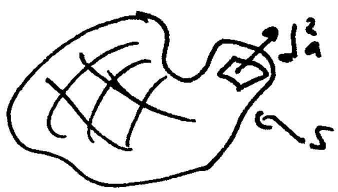
$$\int_V d^3x' \left[u \nabla^2 v - v \nabla^2 u \right] = 0 \quad \text{or}$$

$$u(x) = \int d^3x' \; v(x - x') \rho(x')$$ ← This is just Coulomb's Law from electrostatics!

A slightly amusing application of Gauss's Theorem:

What is the surface integral of a scalar function over a closed surface

$$\oint \varphi \, d^2\vec{a} = ?$$



write a vector field $\vec{w} = \varphi(\vec{x}) \vec{a}$ ← a constant vector.

$$\int_V \vec{\nabla} \cdot \vec{w} \, d^3x = \oint_S \vec{w} \cdot \hat{n} \, d^2a \quad \text{or using} \quad \vec{\nabla} \cdot [a\varphi] = \vec{a} \cdot \vec{\nabla}\varphi$$

$$\vec{a} \cdot \int \vec{\nabla}\varphi \, d^3\vec{x} = \left[\oint \varphi \, d^2\vec{a} \right] \cdot \vec{a} \quad \text{or}$$

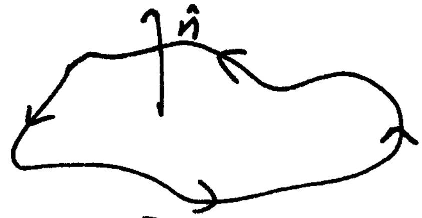
$$\vec{a} \cdot \left[\int \vec{\nabla}\varphi \, d^3x - \oint \varphi \, d^2\vec{a} \right] = 0$$ for any $\vec{a}$. $\Rightarrow$ It must be the term in $[\;] = 0$.

Note: This is a very powerful way to prove results, but it relies on the fact that we have made no claims about the vector $\vec{a}$ ⟷ it is arbitrary.

Thus $\oint \varphi \, d^2\vec{a} = \int \vec{\nabla}\varphi \, d^3x$

Stokes Theorem:
$$\oint_{\partial A} \vec{V}\cdot d\vec{s} = \int_{A} (\vec{\nabla}\times\vec{V})\cdot d^2\vec{a}$$

line integral around the boundary of A.



Can you prove this one from p.17? Try to make an argument like we did for Gauss's law. Also see Arfkin p. 61.

Potential Theory, Exact Differentials and Thermodynamics

If a vector field can be expressed as the gradient of a potential $\vec{F} = -\vec{\nabla}\varphi$, the field is called conservative. Explain why the minus sign is tradiational.

Why is this called conservative?

(1) $\vec{\nabla}\times\vec{F} = 0$

(2) $\oint \vec{F}\cdot d\vec{r} = 0$ for <u>any</u> closed path.

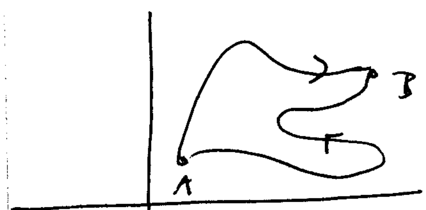
1st note $\vec{\nabla}\times\vec{\nabla}\varphi = 0$ since

$$[\vec{\nabla}\times\vec{\nabla}\varphi]_\alpha = \underbrace{\varepsilon_{\alpha ij}}_{\text{antisymmetric}} \overbrace{\partial_i \partial_j}^{\text{symmetric}} \varphi = 0$$

Similarly, $-\oint \vec{F}\cdot d\vec{r} = -\oint \vec{\nabla}\varphi \cdot d\vec{r} = -\oint d\varphi = 0$

since φ is single-valued.

Notice this means that the integral of $\vec{F}\cdot d\vec{r}$ between any two points is <u>independent of path</u>



$$\int_{C_1, C_2} \vec{F}\cdot d\vec{r} = \Delta\phi_{AB} = \varphi_B - \varphi_A .$$

Think about a gravitational potential.

Application to thermodynamics: Exact and inexact differentials

$df = P(x,y)\,dx + Q(x,y)\,dy$ ⟵ Is this a "real" i.e. exact differential or not? Consider $đW = -p\,dV$ ⟵ an inexact differential or $dE = T\,ds - p\,dV$ ⟵ an exact differential

To figure this out we need to decide if df is the differential of a function f. In other words

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ ⟵ Thus, because of the equality of the mixed partial derivatives : $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

we must have:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \longleftarrow$$ for df to really be the differential of a function φ.

Note that if we think of this as a vector field:

$$d\vec{F} = P(x,y)\,dx\,\hat{x} + Q(x,y)\,dy\,\hat{y} \quad \text{or}$$

$d\vec{F} = \left(P\hat{x} + Q\hat{y}\right)\cdot d\vec{r}$ then being an exact differential means that

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \text{or} \quad \vec{\nabla}\times\vec{F} = 0.$$

So: exact differential $\Longleftrightarrow$ curl-free vector field